

OPTIMAL PRODUCTION IN MONOPOLY PRICING: A STOCHASTIC AND DYNAMIC APPROACH

Francisco Venegas-Martínez*

Finance Department, Monterrey Institute of Technological and Higher Education,
Mexico City Campus

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Abstract

This paper is concerned with a special case of stochastic linear regulator that arises in optimal production planning in monopoly pricing. The state process consists of the inventory level and the demand rate (the latter taken as given). The inventory level involves both the production rate (control variable) and the demand rate. Several stochastic models for the demand are proposed and discussed. The state equation is a linear controlled stochastic equation driven by a Brownian motion. Costs are assumed to be quadratic and the LQG (Linear-Quadratic-Gaussian) approach is used to characterize optimal production-inventory policies. Finally, when possible, closed-form solutions are provided, otherwise approximate solutions are found by means of discrete-time linear systems.

Resumen

Este artículo se refiere a un caso especial de regulador lineal estocástico que surge en la planeación óptima de la producción en mercados monopólicos. El proceso de estado consiste en el nivel del inventario y la demanda (ésta última tomada como dada). A su vez, el nivel del inventario considera la producción (variable del control) y la demanda. Varios modelos estocásticos del comportamiento de la demanda se proponen y se discuten. La ecuación de estado es una ecuación lineal estocástica controlada, la cual es conducida por un movimiento Browniano. Los costos se suponen cuadráticos y el modelo LQG (Linear-Quadratic-Gaussian) se utiliza para caracterizar las políticas óptimas de producción-inventario. Finalmente, se proporcionan soluciones analíticas cuando sea posible. En caso contrario, se proporcionan soluciones aproximadas a través del uso de sistemas lineales discretos.

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* Instituto Tecnológico y de Estudios Superiores de Monterrey, Campus Ciudad de México, Departamento de Finanzas, Calle del Puente 222, 14380 México, D. F., Telephone (52)54831861, E-mail: fvenegas@campus.ccm.itesm.mx

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1. Introduction

Determining production-inventory policies that provide minimum costs when a stochastic demand is taken as given, is of great importance in studying monopoly pricing. The literature regarding this field is scarce because of the presence of randomness, which introduces technical complexities. In this paper, the demand is assumed to be driven by a continuous Markovian processes with continuous paths (diffusion Markovian processes)¹. Once a quadratic functional expressing production and inventory costs has been defined, the LQG approach is then used to characterize optimal production-inventory policies. Closed-form and approximate solutions, for several stochastic demand models taken as given, are found².

This paper is organized as follows. In section 2, we propose several stochastic models for the demand. Under this framework, we study an oligopoly market with several plants operating jointly. In section 3, we present a family of quadratic functionals expressing the inventory and production costs. It is here where the LQG approach is used to characterize optimal solutions. Through section 4, we provide closed-form and approximate solutions. Finally, in section 5, we give a set of comments and remarks, as well as delimitations of the results obtained.

2. Basic Demand Models

The following models take into account the production of a single good and finite horizon. The multidimensional case for the production of several goods with independent demands will be treated later through subsections 2.3 and 2.4. The starting point of our analysis is the so-called balance equation:

$$\sigma[I(t) - i_0] = \int_0^t [(\sigma U(s) + \mu) - X(s)] ds, \quad (2.1)$$

with the conditions $X(0) = x_0$, $I(0) = i_0$, $X(t) \geq \mu$, $0 \leq t \leq T$. Here, $X(t)$ is the rate of demand at time t ; μ (the physical drift) is mean expected rate of demand; σ is the constant diffusion coefficient or instantaneous volatility; $\sigma U(t) + \mu$ is the rate of production, scaled by σ , at time t , which is seen as a control function; and $\sigma I(t)$ is the level of inventory at time t , scaled by σ , too.

2.1 First-Order Demand Models

We first assume that the demand, $X(t)$, satisfies the first order stochastic differential equation of exponential growth (*cf.* Bensoussan *et al.* 1984).

$$d\left(\frac{X(t) - \mu}{\sigma}\right) = -a\left(\frac{X(t) - \mu}{\sigma}\right) dt + dZ(t), \quad 0 \leq t \leq T, \quad X(0) = x_0, \quad (2.2)$$

¹ Problems in which the supply has random disturbances, for instance, the determination of optimal operation policies in dams, are studied in Hernández (1987), Pliska (1975), and Bather (1968).

² As related references for the LQG approach we may mention, for instance: Arnold (1982), Elliott (1982), Bensoussan (1982), and Fleming and Rishel (1975).

where a is a constant different from zero and $Z(t) \sim \mathcal{N}(0, dt)$ is a standard Wiener process (standard Brownian motion, the continuous time version of a random walk). It is through the constant a that it is possible to model a demand trend, increasing, for $-a > 0$, or decreasing, for $-a < 0$. The term $dZ(t)$ appearing in (2.2) produces an effect of stochastic disturbances in the behavior of the demand. We may think of $X(t) - \mu$ in (2.2) as a process with average growth rate $-a$ and variance σ^2 . By applying elementary stochastic calculus, it can be readily shown that the solution of (2.2) is given by

$$X(t) = \mu + e^{-at}(x_0 - \mu) + \int_0^t e^{-a(t-s)} \sigma dZ(s).$$

Notice now that equations (2.1) and (2.2) can be written together in matrix form as

$$dY(t) = [AY(t) + BV(t)]dt + CdZ(t), \quad Y(0) = y_0, \quad (2.3)$$

by setting

$$Y(t) = \begin{pmatrix} X(t) - \mu \\ \sigma I(t) \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 \\ U(t) \end{pmatrix}, \quad A = \begin{pmatrix} -a & 0 \\ -1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \quad C = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} x_0 - \mu \\ \sigma i_0 \end{pmatrix}.$$

Equation (2.3) is known as the equation of evolution of the linear regulator (see, for instance, Fleming and Rishel, 1975). This equation plays an important role in many practical situations, mainly, in finance and economics.

2.2 Second-Order Demand Models

If we have an occurrence frequency between periods of high and low demand in $[0, T]$, then we may assume that the demand $X(t)$ satisfies the following second-order stochastic differential equation (the stochastic harmonic oscillator):

$$\frac{d^2}{dt^2} \left(\frac{X(t) - \mu}{\sigma} \right) = -a \left(\frac{X(t) - \mu}{\sigma} \right) + \frac{dZ(t)}{dt}, \quad 0 \leq t \leq T, \quad (2.4)$$

with initial conditions $X(0) = x_0$, and $\dot{X}(0) = \xi_0$. Here, the trend a is a positive constant. The other constants μ and σ are taken again as in equation(2.1). We warn the reader about the term $dZ(t)/dt$ in the above expression, this derivative does not exist and should be adequately interpreted as stochastic disturbances. By using elementary stochastic calculus, it can be shown that

$$X(t) = \mu + \int_0^t e^{-as}(x_0 - \mu)ds + \int_0^t \int_0^u e^{-a(u-s)} \sigma dZ(s)du$$

is solution of (2.4). Notice that, with a suitable change of variable, equation (2.4) can be written as

$$\begin{cases} d(X(t) - \mu) = \sigma L(t)dt, \\ d\sigma L(t) = -a(X(t) - \mu)dt + \sigma dZ(t), \\ X(0) = x_0, \quad L(0) = (1/\sigma)\xi_0. \end{cases} \quad (2.5)$$

In this case, equations (2.1) and (2.5) may be set in similar form to (2.3), by taking

$$Y(t) = \begin{pmatrix} X(t) - \mu \\ \sigma I(t) \\ \sigma L(t) \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 \\ U(t) \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}, \quad y_0 = \begin{pmatrix} x_0 - \mu \\ \sigma i_0 \\ \xi_0 \end{pmatrix}.$$

Suppose now that both frequency and demand decrease in time, then we may assume that the demand satisfies the equation for the stochastic damped oscillator:

$$\frac{d^2}{dt^2} \left(\frac{X(t) - \mu}{\sigma} \right) = -c \frac{d}{dt} \left(\frac{X(t) - \mu}{\sigma} \right) - a \left(\frac{X(t) - \mu}{\sigma} \right) + \frac{dZ(t)}{dt}, \quad (2.6)$$

with conditions $X(0) = x_0$ and $\dot{X}(0) = \xi_0$. Here, c is a positive constant. In this case, taking the same change of variable as in (2.5), we get the following expression:

$$\begin{cases} d(X(t) - \mu) = \sigma L(t)dt, \\ d\sigma L(t) = -c\sigma L(t)dt - a(X(t) - \mu)dt + \sigma dZ(t), \\ X(0) = x_0, \quad L(0) = (1/\sigma)\xi_0. \end{cases} \quad (2.7)$$

For this model, instead of having the matrix A , given before, we now have the matrix

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -a & 0 & -c \end{pmatrix}.$$

We shall refer in what follows to (2.4) as the second-order model for A , and to (2.6) as the second-order model for A_0 .

Finally, we present a model for coupled demands $X_1(t)$ and $X_2(t)$, both following second-order models, by assuming they satisfy the system:

$$\begin{aligned} \frac{d^2}{dt^2} \left(\frac{X_1(t) - \mu_1}{\sigma_1} \right) &= -a_1 \left(\frac{X_1(t) - \mu_1}{\sigma_1} \right) + a \left[\beta \left(\frac{X_2(t) - \mu_2}{\sigma_2} \right) \right. \\ &\quad \left. - \left(\frac{X_1(t) - \mu_1}{\sigma_1} \right) \right] + \frac{d}{dt} Z_1(t), \\ \frac{d^2}{dt^2} \left(\frac{X_2(t) - \mu_2}{\sigma_2} \right) &= -a_2 \left(\frac{X_2(t) - \mu_2}{\sigma_2} \right) - a \left[\left(\frac{X_2(t) - \mu_2}{\sigma_2} \right) \right. \\ &\quad \left. - \beta^{-1} \left(\frac{X_1(t) - \mu_1}{\sigma_1} \right) \right] + \frac{d}{dt} Z_2(t). \end{aligned}$$

with initial conditions $X_1(0) = x_{1,0}$, $X_2(0) = x_{2,0}$, $\dot{X}_1(0) = \xi_{1,0}$, $\dot{X}_2(0) = \xi_{2,0}$. The constants a , a_1 , a_2 and β are all positive. Moreover, the constant β is a technical coefficient that may be interpreted as the number of units of X_1 required to make a unit of X_2 . The processes $Z_1(t)$ and $Z_2(t)$ are assumed to be independent. We may obtain a similar equation to that of (2.3) by setting

$$\left\{ \begin{array}{l} d\sigma_1 I_1(t) = [\sigma_1 U_1(t) - (X_1(t) - \mu_1)]dt, \quad X_1(t) \geq \mu_1, \\ d\sigma_2 I_2(t) = [\sigma_2 U_2(t) - (X_2(t) - \mu_2)]dt, \quad X_2(t) \geq \mu_2, \\ d(X_1(t) - \mu_1) = \sigma_1 L_1(t)dt, \\ d(X_2(t) - \mu_2) = \sigma_2 L_2(t)dt, \\ d\sigma_1 \sigma_2 L_1(t) = -(a + a_1)\sigma_2(X_1(t) - \mu_1)dt + a\sigma_1\beta(X_2(t) - \mu_2)dt \\ \quad + \sigma_1 \sigma_2 dZ_1(t), \\ d\sigma_1 \sigma_2 L_2(t) = -(a + a_2)\sigma_1(X_2(t) - \mu_2)dt + a\sigma_2\beta^{-1}(X_1(t) - \mu_1)dt \\ \quad + \sigma_1 \sigma_2 dZ_2(t), \\ X_k(0) = x_{k,0}, \quad I_k(0) = i_{k,0}, \quad L_k(0) = (1/\sigma_k)\xi_{k,0}, \quad k = 0, 1, \end{array} \right.$$

and choosing

$$Y(t) = \begin{pmatrix} X_1(t) - \mu_1 \\ X_2(t) - \mu_2 \\ \sigma_1 I_1(t) \\ \sigma_2 I_2(t) \\ \sigma_1 \sigma_2 L_1(t) \\ \sigma_1 \sigma_2 L_2(t) \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 \\ 0 \\ U_1(t) \\ U_2(t) \\ 0 \\ 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} x_{1,0} - \mu_1 \\ x_{2,0} - \mu_2 \\ \sigma_1 i_{1,0} \\ \sigma_2 i_{2,0} \\ \sigma_2 \xi_{1,0} \\ \sigma_1 \xi_{2,0} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \sigma_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_1^{-1} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -(a + a_1)\sigma_2 & a\sigma_1\beta & 0 & 0 & 0 & 0 \\ a\sigma_2\beta^{-1} & -(a + a_2)\sigma_1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

B a matrix with entries $B_{33} = \sigma_1$, $B_{44} = \sigma_2$, $B_{ij} = 0$, otherwise, and C a matrix with entries $C_{51} = \sigma_1\sigma_2$, $C_{62} = \sigma_1\sigma_2$, $C_{ij} = 0$, otherwise. In this case, $Z(t)$ is a vector of independent Wiener processes, containing as the first two components $Z_1(t)$ and $Z_2(t)$.

2.3 A Monopoly Model with Production Plants Operating Jointly

For the sake of simplicity, we first consider only two plants operating jointly; the ideas involved in the proposed model can be easily extended to more plants. Assume that plants P_1 and P_2 produce the same generic good, and that P_1 has greater capability of production than P_2 . Suppose also that from the total production of P_1 , $U_1(t)$, a constant proportion, say $(1 - \alpha)U_1(t)$, for a fixed $\alpha \in (0, 1)$, is transferred to plant P_2 , so that the latter is able to satisfy its

demand. If $X_1(t)$ and $X_2(t)$ stand for the demands at each plant. Then the balance equations turn out to be:

$$\begin{cases} d\sigma_1 I_1(t) = [\sigma_1 \alpha U_1(t) - (X_1(t) - \mu_1)]dt, & X_1(t) \geq \mu_1, \\ d\sigma_2 I_2(t) = \{\sigma_2 [(1 - \alpha)U_1(t) + U_2(t)] - (X_2(t) - \mu_2)\}dt, & X_2(t) \geq \mu_2, \\ X_1(0) = x_{1,0}, \quad X_2(0) = x_{2,0}, \quad I_1(0) = i_{1,0}, \quad I_2(0) = i_{2,0}. \end{cases} \tag{2.8}$$

Assuming now that $X_1(t)$ and $X_2(t)$ are independent processes and that, for instance, they both follow a first-order model with coefficients a and b , respectively, then taking

$$Y(t) = \begin{pmatrix} X_1(t) - \mu_1 \\ X_2(t) - \mu_2 \\ \sigma_1 I_1(t) \\ \sigma_2 I_2(t) \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 \\ 0 \\ U_1(t) \\ U_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\sigma_1 & 0 \\ 0 & 0 & (1 - \alpha)\sigma_2 & \sigma_2 \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$y_0 = \begin{pmatrix} x_{1,0} - \mu_1 \\ x_{2,0} - \mu_2 \\ \sigma_1 i_{1,0} \\ \sigma_2 i_{2,0} \end{pmatrix},$$

we get a similar expression to (2.3) with $Z(t) = [Z_1(t), Z_2(t), Z_3(t), Z_4(t)]'$, (an apostrophe will denote the transpose operation), is a vector of independent Wiener processes. The processes $Z_3(t)$ and $Z_4(t)$ have been merely introduced for technical convenience.

The previous arguments have been applied when both plants produce the same good. However, when the plants produce different goods, say, g_1 and g_2 , so that g_1 is required to produce g_2 , then we just need to multiply the transferred quantity by a technical coefficient β to transform units of g_1 into units of g_2 . Thus, either assuming that the transformed production, $\beta(1 - \alpha)U_1(t)$, is additional to some production $U_2(t)$, which is obtained by P_2 without using resources from P_1 , we have that the balance equation for P_2 is

$$d\sigma_2 I_2(t) = \{\sigma_2 [\beta(1 - \alpha)U_1(t) + U_2(t)] - (X_2(t) - \mu_2)\}dt,$$

or assuming that all the production of P_2 is the transformed production, then the balance equation for P_2 simple becomes

$$d\sigma_2 I_2(t) = \{\sigma_2 \beta(1 - \alpha)U_1(t) - (X_2(t) - \mu_2)\}dt.$$

2.4 Multi-dimensional Demand Models

Consider a demand vector for n goods, say, $X(t) = [X_1(t), X_2(t), \dots, X_n(t)]'$, which consists of independent components. In this case, the system of balance equations and the multivariate first-order model have the following extensions:

$$\begin{cases} d\Sigma I(t) = [\Sigma U(t) - (X(t) - \mu)]dt, \\ d(X(t) - \mu) = -\Lambda(X(t) - \mu) + \Sigma dZ(t), \\ X(0) = x_0, \quad I(0) = i_0, \end{cases} \tag{2.9}$$

where $\Sigma = [\sigma_i]_{i=1}^n$, $\sigma_i > 0$, $i = 1, 2, \dots, n$, and $\Lambda = [\lambda_i]_{i=1}^n$, $\lambda_i \neq 0$, $i = 1, 2, \dots, n$, are diagonal matrices; $Z(t)$ is a random vector containing independent Wiener processes; $\Sigma U(t) + \mu$ is the vector of scaled productions; and $\Sigma I(t)$ is the vector of scaled inventories. The (unique) solution, in vector form, of the stochastic differential equation in (2.9) is given by

$$X(t) = \mu + e^{-\Lambda t}(x_0 - \mu) + \int_0^t e^{-\Lambda(t-s)} \Sigma dZ(s).$$

According to the extension of (2.3) to the multidimensional case, we have that the dimension of the corresponding vector $Y(t) = [X(t) - \mu, \Sigma I(t)]'$ is $2n$. The corresponding matrices A , B , and C follow a similar pattern to that of the matrices from subsection 2.3.

The analyses for the case in which the demand model is of second order, and even the mixed case (first and second-order models interacting) have no essential differences from the analysis done before, since components are individually treated.

3. Quadratic Cost Functionals: LQG Theory

Next, we introduce a family of quadratic functionals to evaluate the total cost of production and inventory, when an initial condition $Y(t) = y$ is given and a control (production) function $V(y, t)$ is used. The family of functionals with which we shall be concerned, $\{J_{P,Q,M}^V\}$, is a function of matrices P , Q and M , where P and M are both symmetric positive semidefinite, and Q is symmetric positive definite. These matrices keep technical coefficients for costs. In practice, very often, these costs have a quadratic behavior, (*e.g.* costs of generation of electric energy), being proportional to the functional

$$\begin{aligned} J_{P,Q,M}^V(y, T - t) = E \left\{ \int_t^T [\langle Y(s), PY(s) \rangle + \langle V(s), QV(s) \rangle] ds \mid \mathfrak{F}_{y,V} \right\} \\ + \langle Y(T), MY(T) \rangle, \end{aligned} \tag{3.1}$$

where

$$Y(s) = y_0 + \int_0^s [AY(u) + BV(u)]du + \int_0^s C dZ(u),$$

here, $\langle \cdot, \cdot \rangle$ is the usual inner product in the Euclidean space \mathbb{R}^n , $\mathfrak{S}_{y,V}$ denotes the sets $\{Y(t) = y \text{ is given, and the control } V(y, t) \text{ is used}\}$, and $\langle Y(T), MY(T) \rangle$ is a terminal cost.

We notice that the integral in (3.1) takes a stochastic behavior because of its relation with the linear regulator. From now on, we shall leave out the subindexes P , Q and M , and denote $\tau = T - t = \text{“time to go”}$, $0 \leq \tau \leq T$ (so $t = \text{“absolute time”}$). The LQG approach characterizes $V^*(y, t)$, so that

$$J(y, \tau) = J^{V^*}(y, \tau) = \min_{\substack{V(y,s) \\ t \leq s \leq T}} J^V(y, \tau), \tag{3.2}$$

where the minimum is taken over all admissible controls $V(y, s)$ in $[t, T]$ (*v.g.* controls satisfying Lipschitz conditions uniformly in the variable y). Therefore, the LQG approach leads us to

$$V^*(y, t) = -Q^{-1}B'S^*(\tau)y, \tag{3.3}$$

where $S^*(\tau)$ is a symmetric positive definite matrix which solves

$$-\frac{d}{d\tau}S(\tau) + A'S(\tau) + S(\tau)A + P - S(\tau)BQ^{-1}B'S(\tau) = 0, \tag{3.4}$$

with the condition $S(0) = M$. Equation (3.4) is known as the Ricatti differential equation, which may be seen as the adjoint equation of the optimal trajectory $S(\tau)$. A meaningful fact about this equation is that, under very general conditions, is that it has a bounded solution on any finite interval (see, for instance, Fleming and Rishel, 1975).

4. Optimal Characterization of Production-Inventory Policies

In this section, as immediate consequence of (3.3), we shall characterize optimal production-inventory policies for the discussed models, when the cost technical coefficients for P , Q , and M do not include crossed terms, that is, when $Q = M = I$ and P is a diagonal matrix with non-negative terms. When P , Q and M are arbitrary diagonal matrices, the analysis will be basically the same. The general case, when P , Q and M are taken as in (3.1), will just take a little more of effort, making the notation cumbersome.

4.1 Optimal Production-Inventory Policies for the First-Order Demand Model

For the first-order model introduced in subsection 2.1, if we take $Q = M = I$ and $P_{22} = 1$, $P_{ij} = 0$, otherwise, it follows that the optimal control satisfies

$$U^*(y, t) = -\sigma[S_{21}^*(\tau)y_1 + S_{22}^*(\tau)y_2], \quad Y(t) = y, \quad \tau = T - t,$$

where $S_{21}^*(\tau)$ and $S_{22}^*(\tau)$ are solutions of the system of differential equations

$$\begin{cases} \frac{d}{d\tau} S_{21}(\tau) = -[a + \sigma^2 S_{22}(\tau)] S_{21}(\tau) - S_{22}(\tau), \\ \frac{d}{d\tau} S_{22}(\tau) = 1 - \sigma^2 S_{22}^2(\tau), \quad S_{22}(0) = 1, \quad S_{21}(0) = 0. \end{cases} \quad (4.1)$$

The solution of system (4.1), provided $\sigma \neq 1$, is given by

$$\begin{cases} S_{22}^*(\tau) = \frac{1}{\sigma} \left[\frac{(1 + \sigma)(1 - \sigma)^{-1} e^{2\sigma\tau} - 1}{(1 + \sigma)(1 - \sigma)^{-1} e^{2\sigma\tau} + 1} \right], \\ S_{21}^*(\tau) = - \int_0^\tau S_{22}^*(s) \exp \{-a(\tau - s) - [Q(\tau) - Q(s)]\} ds, \end{cases} \quad (4.2)$$

where

$$Q(\tau) = \sigma^2 \int_0^\tau S_{22}^*(u) du. \quad (4.3)$$

When $\sigma = -1$, we get $S_{22}^*(\tau) \equiv -1$. In this case, $S_{21}^*(\tau) = (1 - a)^{-1}(1 - e^{-\tau(1-a)})$ for $a \neq 1$, and $S_{21}^*(\tau) = -\tau$ for $a = 1$.

We may provide an approximate solution for $S_{21}(\tau)$, when τ is small, as follows: observe that, if τ is small, so is $\tau - s$, since $0 \leq s \leq \tau$. Therefore, $Q(\tau) - Q(s) \approx Q'(\tau)(\tau - s)$, then, for small τ , we may well use the approximation

$$S_{21}^*(\tau) \approx - \int_0^\tau S_{22}^*(s) e^{-(\tau-s)[a + \sigma^2 S_{22}^*(\tau)]} ds.$$

4.2 Optimal Production-Inventory Policies for the Second-Order Demand Models

When the demand follows the second-order model for $A, Q = M = I$ and $P_{22} = 1, P_{ij} = 0$, otherwise, the optimal control has the form

$$U^*(y, t) = -\sigma[S_{21}^*(\tau)y_1 + S_{22}^*(\tau)y_2 + S_{23}^*(\tau)y_3], \quad Y(t) = y, \quad \tau = T - t, \quad (4.4)$$

where $S_{22}^*(\tau)$ is taken again as in (4.2), and $(S_{21}^*(\tau), S_{23}^*(\tau))$ is solution of the system

$$\begin{cases} \frac{d}{dt} S_{21}(\tau) = -[1 + \sigma^2 S_{21}(\tau)] S_{22}^*(\tau) - a S_{23}(\tau), \\ \frac{d}{dt} S_{23}(\tau) = S_{21}(\tau) - \sigma^2 S_{22}^*(\tau) S_{23}(\tau), \\ S_{21}(0) = S_{23}(0) = 0. \end{cases} \quad (4.5)$$

If we suppose a parameter value $a = 1$, the joint solution of (4.5) turns out to be (see, for instance, Hirsh 1974, ch. 5):

$$\begin{cases} S_{21}^*(\tau) = - \int_0^\tau S_{22}^*(s) e^{-[Q(\tau) - Q(s)]} \cos(\tau - s) ds, \\ S_{23}^*(\tau) = - \int_0^\tau S_{22}^*(s) e^{-[Q(\tau) - Q(s)]} \sin(\tau - s) ds, \end{cases}$$

where we have taken $Q(\tau)$ as in (4.3). Approximate solutions for $S_{21}(\tau)$ and $S_{23}(\tau)$, when τ is small, can be found by using $\sin(\tau - s) \approx \tau - s$ and $\cos(\tau - s) \approx 1 - (1/2)(\tau - s)^2$, so

$$\begin{cases} S_{21}^*(\tau) \approx - \int_0^\tau S_{22}^*(s) e^{-(\tau-s)\sigma^2 S_{22}^*(\tau)} \left[1 - \frac{1}{2}(\tau - s)^2\right] ds, \\ S_{23}^*(\tau) \approx - \int_0^\tau S_{22}^*(s) e^{-(\tau-s)\sigma^2 S_{22}^*(\tau)} (\tau - s) ds. \end{cases}$$

Next, we shall provide approximate solutions for the second-order model for A_0 , the optimal control also satisfies (4.4), but $[S_{21}^*(\tau), S_{22}^*(\tau), S_{23}^*(\tau)]$ are now solutions of the system

$$\begin{pmatrix} S_{21}(\tau) \\ S_{23}(\tau) \end{pmatrix} = H^*(\tau) \begin{pmatrix} S_{21}(\tau) \\ S_{23}(\tau) \end{pmatrix} - \begin{pmatrix} S_{22}^*(\tau) \\ 0 \end{pmatrix}, \tag{4.6}$$

where

$$H^*(\tau) = \begin{pmatrix} -\sigma^2 S_{22}^*(\tau) & -a \\ 1 & -c - \sigma_2 S_{22}^*(\tau) \end{pmatrix},$$

and $S_{22}^*(\tau)$ is taken as in (4.2).

In order to obtain approximate solutions of (4.6), we proceed to apply Euler's method, getting the following discrete-time linear system:

$$\begin{pmatrix} S_{21}(\tau_{k+1}) \\ S_{23}(\tau_{k+1}) \end{pmatrix} = \left[\frac{T}{N} H^*(\tau_k) + I \right] \begin{pmatrix} S_{21}(\tau_k) \\ S_{23}(\tau_k) \end{pmatrix} - \frac{T}{N} \begin{pmatrix} S_{22}^*(\tau_k) \\ 0 \end{pmatrix}, \tag{4.7}$$

where $\tau_k = k \frac{T}{N}$, $k = 0, 1, 2, \dots, N$, and the matrix $H^*(\tau_k)$ is given by

$$H^*(\tau_k) = \begin{pmatrix} -\sigma^2 S_{22}^*(\tau_k) & -a \\ 1 & -c - \sigma_2 S_{22}^*(\tau_k) \end{pmatrix}.$$

The solution of (4.7) is found as

$$\begin{pmatrix} S_{21}^*(\tau_{k+1}) \\ S_{23}^*(\tau_{k+1}) \end{pmatrix} = -\frac{T}{N} \left\{ \prod_{i=1}^k \left(\frac{T}{N} H^*(\tau_i) + I \right) \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{T}{N} \begin{pmatrix} S_{22}^*(\tau_k) \\ 0 \end{pmatrix} - \frac{T}{N} \sum_{i=1}^k \left[\left(\frac{T}{N} H^*(\tau_k) + I \right) \cdots \left(\frac{T}{N} H^*(\tau_{i+1}) + I \right) \right] \begin{pmatrix} S_{22}^*(\tau_i) \\ 0 \end{pmatrix}.$$

Finally, we consider the model for coupled demands when $Q = M = I$ and $P_{33} = P_{44} = 1$, $P_{ij} = 0$, otherwise. In this case, the optimal controls are given by

$$\begin{cases} U_1^*(y, t) = -\sigma_1 \sum_{j=1}^4 S_{3j}^*(\tau) y_j, \\ U_2^*(y, t) = -\sigma_2 \sum_{j=1}^4 S_{4j}^*(\tau) y_j, \end{cases}$$

where $Y(t) = y$ and $\tau = T - t$. Here, $[S_{3j}^*]_{j=1}^4$ and $[S_{4j}^*]_{j=1}^4$ are the joint solution of the system of differential equations, which, for convenience, has been separated into three groups of equations, as follows:

$$\left\{ \begin{aligned} \frac{d}{d\tau} S_{33}(\tau) &= 1 - \sigma_1^2 S_{33}^2(\tau) - \sigma_2^2 S_{43}^2(\tau), \\ \frac{d}{d\tau} S_{44}(\tau) &= 1 - \sigma_1^2 S_{43}^2(\tau) - \sigma_2^2 S_{44}^2(\tau), \\ \frac{d}{d\tau} S_{43}(\tau) &= -\sigma_1^2 S_{43}(\tau) S_{33}(\tau) - \sigma_2^2 S_{44}(\tau) S_{43}(\tau), \\ \\ \frac{d}{d\tau} S_{31}(\tau) &= -S_{33}(\tau) - (a + a_1)\sigma_2 S_{35}(\tau) + a\sigma_2\beta^{-1} S_{36}(\tau) - \sigma_1^2 S_{31}(\tau) S_{33}(\tau) \\ &\quad - \sigma_2^2 S_{41}(\tau) S_{43}(\tau), \\ \frac{d}{d\tau} S_{32}(\tau) &= -S_{34}(\tau) + a\sigma_1\beta S_{35}(\tau) - (a + a_2)\sigma_1 S_{36}(\tau) - \sigma_1^2 S_{32}(\tau) S_{33}(\tau) \\ &\quad - \sigma_2^2 S_{42}(\tau) S_{43}(\tau), \\ \frac{d}{d\tau} S_{35}(\tau) &= \sigma_2^{-1} S_{31}(\tau) - \sigma_1^2 S_{33}(\tau) S_{35}(\tau) - \sigma_2^2 S_{45}(\tau) S_{43}(\tau), \\ \frac{d}{d\tau} S_{36}(\tau) &= \sigma_1^{-1} S_{32}(\tau) - \sigma_1^2 S_{36}(\tau) S_{33}(\tau) - \sigma_2^2 S_{46}(\tau) S_{43}(\tau), \\ \\ \frac{d}{d\tau} S_{41}(\tau) &= -S_{34}(\tau) - (a + a_1)\sigma_2 S_{45}(\tau) + a\sigma_2\beta^{-1} S_{46}(\tau) - \sigma_1^2 S_{31}(\tau) S_{34}(\tau) \\ &\quad - \sigma_2^2 S_{41}(\tau) S_{44}(\tau), \\ \frac{d}{d\tau} S_{42}(\tau) &= -S_{44}(\tau) - a\sigma_2\beta S_{45}(\tau) - (a + a_2)\sigma_1 S_{46}(\tau) - \sigma_1^2 S_{32}(\tau) S_{34}(\tau) \\ &\quad - \sigma_2^2 S_{42}(\tau) S_{44}(\tau), \\ \frac{d}{d\tau} S_{45}(\tau) &= \sigma_2^{-1} S_{41}(\tau) - \sigma_1^2 S_{35}(\tau) S_{34}(\tau) - \sigma_2^2 S_{45}(\tau) S_{44}(\tau), \\ \frac{d}{d\tau} S_{46}(\tau) &= \sigma_1^{-1} S_{42}(\tau) - \sigma_1^2 S_{36}(\tau) S_{34}(\tau) - \sigma_2^2 S_{46}(\tau) S_{44}(\tau), \end{aligned} \right.$$

with the conditions $S_{33}(0) = S_{44}(0) = 1$, and $S_{31}(0) = S_{32}(0) = S_{41}(0) = S_{42}(0) = S_{43}(0) = 0$. We shall provide solutions when $\sigma_1 = \sigma_2 = \sigma$. In this case, $S_{33} \equiv S_{44}$, and the first three-equation group becomes

$$\left\{ \begin{aligned} \frac{d}{d\tau} S_{33}(\tau) &= 1 - \sigma^2 [S_{33}^2(\tau) + S_{43}^2(\tau)], \\ \frac{d}{d\tau} S_{43}(\tau) &= -2\sigma^2 S_{33}(\tau) S_{43}(\tau), \end{aligned} \right.$$

from where

$$\left\{ \begin{aligned} \frac{d}{d\tau} [S_{33}(\tau) + S_{43}(\tau)] &= 1 - \sigma^2 [S_{33}(\tau) + S_{43}(\tau)]^2, \\ \frac{d}{d\tau} [S_{33}(\tau) - S_{43}(\tau)] &= 1 - \sigma^2 [S_{33}(\tau) - S_{43}(\tau)]^2, \end{aligned} \right.$$

so that, $S_{33} + S_{34} \equiv S_{33} - S_{34}$, therefore $S_{34}^* \equiv 0$ and so S_{33}^* and S_{44}^* both coincide with S_{22}^* in (4.2). The second four-equation group may be written now as

$$\frac{d}{dt} \begin{pmatrix} S_{31}(\tau) \\ S_{32}(\tau) \\ S_{35}(\tau) \\ S_{36}(\tau) \end{pmatrix} = [-\sigma^2 S_{33}^*(\tau) I + K] \begin{pmatrix} S_{31}(\tau) \\ S_{32}(\tau) \\ S_{35}(\tau) \\ S_{36}(\tau) \end{pmatrix} + \begin{pmatrix} -S_{33}^*(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.8)$$

where

$$K = \begin{pmatrix} 0 & 0 & -(a + a_1)\sigma & a\sigma\beta^{-1} \\ 0 & 0 & a\sigma\beta & -(a + a_2)\sigma \\ \sigma^{-1} & 0 & 0 & 0 \\ 0 & \sigma^{-1} & 0 & 0 \end{pmatrix}.$$

By applying Euler's method to (4.8), we obtain the following discrete-time linear system, with $\tau_k = k(T/N)$, $k = 0, 1, 2, \dots, N$:

$$\begin{pmatrix} S_{31}(\tau_{k+1}) \\ S_{32}(\tau_{k+1}) \\ S_{35}(\tau_{k+1}) \\ S_{36}(\tau_{k+1}) \end{pmatrix} = \left[\frac{T}{N}(K - \sigma^2 S_{33}^*(\tau_k)) + I \right] \begin{pmatrix} S_{31}(\tau_k) \\ S_{32}(\tau_k) \\ S_{35}(\tau_k) \\ S_{36}(\tau_k) \end{pmatrix} - \frac{T}{N} S_{33}^*(\tau_k) \mathbf{e}_1, \quad (4.9)$$

The solutions of (4.9) is

$$\begin{pmatrix} S_{31}^*(\tau_{k+1}) \\ S_{32}^*(\tau_{k+1}) \\ S_{35}^*(\tau_{k+1}) \\ S_{36}^*(\tau_{k+1}) \end{pmatrix} = -\frac{T}{N} \left\{ \prod_{i=1}^k \left[\frac{T}{N}(K - \sigma^2 S_{33}^*(\tau_i)) + I \right] \right\} \mathbf{e}_1 - \frac{T}{N} S_{33}^*(\tau_k) \mathbf{e}_1 \\ - \frac{T}{N} \sum_{i=1}^k \left\{ \left[\frac{T}{N}(K - \sigma^2 S_{33}^*(\tau_k)) + I \right] \cdots \left[\frac{T}{N}(K - \sigma^2 S_{33}^*(\tau_{i+1})) + I \right] \right\} S_{33}^*(\tau_i) \mathbf{e}_1$$

where $\mathbf{e}_1 = [1, 0, 0, 0]'$. The third four-equation group can be similarly treated.

4.3 Optimal Production-Inventory Policies for the Model of Plants Operating Jointly

For the monopoly model with plants operating jointly, we have that the optimal controls, provided $Q = M = I$ and $P_{33} = P_{44} = 1$, $P_{ij} = 0$, otherwise, satisfy

$$\begin{cases} U_1^*(y, t) = - \left[\alpha \sigma_1 \sum_{j=1}^4 S_{3j}^*(\tau) y_j + (1 - \alpha) \sigma_2 \sum_{j=1}^4 S_{4j}^*(\tau) y_j \right], \\ U_2^*(y, t) = -\sigma_2 \sum_{j=1}^4 S_{4j}^*(\tau) y_j, \end{cases} \quad (4.10)$$

where $Y(t) = y$ and $\tau = T - t$. In this case, $[S_{3j}^*]_{j=1}^4$ and $[S_{4j}^*]_{j=1}^4$ are the joint solution of the following differential equation system (which has been separated into three groups of equations)

$$\left\{ \begin{aligned} \frac{d}{d\tau} S_{33}(\tau) &= 1 - \alpha^2 \sigma_1^2 S_{33}^2(\tau) - 2\alpha(1 - \alpha)\sigma_1\sigma_2 S_{34}(\tau) S_{33}(\tau) \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{34}^2(\tau), \\ \frac{d}{d\tau} S_{44}(\tau) &= 1 - \alpha^2 \sigma_1^2 S_{34}^2(\tau) - 2\alpha(1 - \alpha)\sigma_1\sigma_2 S_{44}(\tau) S_{34}(\tau) \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{44}^2(\tau), \\ \frac{d}{d\tau} S_{34}(\tau) &= -\alpha^2 \sigma_1^2 S_{34}(\tau) S_{33}(\tau) - \alpha(1 - \alpha)\sigma_1\sigma_2 [S_{44}(\tau) S_{33}(\tau) + S_{34}^2(\tau)] \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{44}(\tau) S_{34}(\tau), \\ \frac{d}{d\tau} S_{13}(\tau) &= -a S_{13}(\tau) - S_{33}(\tau) - \alpha^2 \sigma_1^2 S_{33}(\tau) S_{13}(\tau) \\ &\quad - \alpha(1 - \alpha)\sigma_1\sigma_2 [S_{34}(\tau) S_{13}(\tau) + S_{33}(\tau) S_{14}(\tau)] \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{14}(\tau) S_{34}(\tau), \\ \frac{d}{d\tau} S_{14}(\tau) &= -a S_{14}(\tau) - S_{34}(\tau) - \alpha^2 \sigma_1^2 S_{34}(\tau) S_{13}(\tau) \\ &\quad - \alpha(1 - \alpha)\sigma_1\sigma_2 [S_{44}(\tau) S_{13}(\tau) + S_{34}(\tau) S_{14}(\tau)] \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{44}(\tau) S_{14}(\tau), \\ \frac{d}{d\tau} S_{23}(\tau) &= -b S_{23}(\tau) - S_{34}(\tau) - \alpha^2 \sigma_1^2 S_{33}(\tau) S_{23}(\tau) \\ &\quad - \alpha(1 - \alpha)\sigma_1\sigma_2 [S_{34}(\tau) S_{23}(\tau) + S_{33}(\tau) S_{24}(\tau)] \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{34}(\tau) S_{24}(\tau), \\ \frac{d}{d\tau} S_{24}(\tau) &= -b S_{24}(\tau) - S_{44}(\tau) - \alpha^2 \sigma_1^2 S_{34}(\tau) S_{23}(\tau) \\ &\quad - \alpha(1 - \alpha)\sigma_1\sigma_2 [S_{44}(\tau) S_{23}(\tau) + S_{34}(\tau) S_{24}(\tau)] \\ &\quad - [\alpha(\alpha - 2) + 2]\sigma_2^2 S_{44}(\tau) S_{24}(\tau), \end{aligned} \right.$$

with the conditions $S_{33}(0) = S_{44}(0) = 1$, and $S_{14}(0) = S_{24}(0) = S_{34}(0) = S_{13}(0) = S_{23}(0) = 0$. We shall provide solutions $[S_{3j}^*]_{j=1}^4$, $[S_{4j}^*]_{j=1}^4$, in the particular case, for which

$$\alpha^2 \sigma_1^2 = q, \text{ and, } \sigma_2^2 = q[(1 - \alpha)^2 + 1]^{-1}, \sigma_1, \sigma_2 > 0, 0 < \alpha < 1,$$

hold for some $q > 0$. Then, for the first three-equation group, $S_{33} \equiv S_{44}$ on $[0, T]$, having instead of three equations, just the following two:

$$\left\{ \begin{aligned} \frac{d}{d\tau} S_{33}(\tau) &= 1 - q[S_{33}^2(\tau) + S_{34}^2(\tau)] - \frac{2q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} S_{33}(\tau) S_{34}(\tau), \\ \frac{d}{d\tau} S_{34}(\tau) &= -\frac{q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} [S_{33}^2(\tau) + S_{34}^2(\tau)] - 2q S_{33}(\tau) S_{34}(\tau), \end{aligned} \right. \tag{4.11}$$

If we denote $C_0^2 = q - \left[q(1 - \alpha) / \sqrt{(1 - \alpha)^2 + 1} \right]$, then from (4.11), we have

$$\begin{cases} \frac{d}{d\tau} S_{33}(\tau) = 1 - q[S_{33}(\tau) + S_{34}(\tau)]^2 + 2C_0^2 S_{33}(\tau) S_{34}(\tau), \\ \frac{d}{d\tau} S_{34}(\tau) = -\frac{q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} [S_{33}(\tau) + S_{34}(\tau)]^2 - 2C_0^2 S_{33}(\tau) S_{34}(\tau), \end{cases} \quad (4.12)$$

if now $C_1^2 = q + \left[q(1 - \alpha) / \sqrt{(1 - \alpha)^2 + 1} \right]$, then from (4.11), we may also write

$$\begin{cases} \frac{d}{d\tau} S_{33}(\tau) = 1 - q[S_{33}(\tau) - S_{34}(\tau)]^2 - 2C_1^2 S_{33}(\tau) S_{34}(\tau), \\ \frac{d}{d\tau} S_{34}(\tau) = -\frac{q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} [S_{33}(\tau) - S_{34}(\tau)]^2 - 2C_1^2 S_{33}(\tau) S_{34}(\tau). \end{cases} \quad (4.13)$$

Setting $F_1(\tau) = S_{33}(\tau) + S_{34}(\tau)$ and $F_0(\tau) = S_{33}(\tau) - S_{34}(\tau)$, we obtain from (4.12) and (4.13) that

$$\frac{d}{dt} F_k(\tau) = C_k^2 \left[\frac{1}{C_k^2} - F_k^2(\tau) \right], \quad F_k(0) = 1, \quad k = 0, 1. \quad (4.14)$$

The solutions of (4.14) are found as

$$F_k(\tau) = \frac{1}{C_k} \left[\frac{(1 + C_k)(1 - C_k)^{-1} e^{2C_k t} - 1}{(1 + C_k)(1 - C_k)^{-1} e^{2C_k t} + 1} \right], \quad k = 0, 1,$$

from which

$$S_{44}^*(\tau) = S_{33}^*(\tau) = \frac{1}{2} [F_0(\tau) + F_1(\tau)], \quad S_{34}^*(\tau) = \frac{1}{2} [F_1(\tau) - F_0(\tau)].$$

Now, define a matrix $H^*(\tau; c)$, for $c = a, b$, with entries

$$\begin{cases} H_{11}^*(\tau; c) = H_{22}^*(\tau; c) = -c - qS_{33}^*(\tau) - \frac{q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} S_{34}^*(\tau), \\ H_{12}^*(\tau; c) = H_{21}^*(\tau; c) = -\frac{q(1 - \alpha)}{\sqrt{(1 - \alpha)^2 + 1}} S_{33}^*(\tau) - qS_{34}^*(\tau), \end{cases}$$

so that

$$\frac{d}{dt} \begin{pmatrix} S_{13}(\tau) \\ S_{14}(\tau) \end{pmatrix} = H^*(\tau; a) \begin{pmatrix} S_{13}(\tau) \\ S_{14}(\tau) \end{pmatrix} - \begin{pmatrix} S_{33}^*(\tau) \\ S_{34}^*(\tau) \end{pmatrix}, \quad (4.15)$$

and

$$\frac{d}{dt} \begin{pmatrix} S_{23}(\tau) \\ S_{24}(\tau) \end{pmatrix} = H^*(\tau; b) \begin{pmatrix} S_{23}(\tau) \\ S_{24}(\tau) \end{pmatrix} - \begin{pmatrix} S_{34}^*(\tau) \\ S_{44}^*(\tau) \end{pmatrix}. \quad (4.16)$$

Also, define the functions

$$Q^*(\tau; c) = - \int_0^\tau H_{11}^*(u; c) du, \quad R^*(\tau; c) = - \int_0^\tau H_{12}^*(u; c) du, \quad c = a, b.$$

Then, the solutions of (4.15) and (4.16) are found as

$$\left\{ \begin{aligned} S_{13}^*(\tau) &= \int_0^\tau S_{34}^*(s) e^{-[Q^*(\tau;a)-Q^*(s;a)]} \sinh[R^*(\tau;a) - R^*(s;a)] ds \\ &\quad - \int_0^\tau S_{33}^*(s) e^{-[Q^*(\tau;a)-Q^*(s;a)]} \cosh[R^*(\tau;a) - R^*(s;a)] ds, \\ S_{14}^*(\tau) &= \int_0^\tau S_{33}^*(s) e^{-[Q^*(\tau;a)-Q^*(s;a)]} \sinh[R^*(\tau;a) - R^*(s;a)] ds \\ &\quad - \int_0^\tau S_{34}^*(s) e^{-[Q^*(\tau;a)-Q^*(s;a)]} \cosh[R^*(\tau;a) - R^*(s;a)] ds, \\ S_{23}^*(\tau) &= \int_0^\tau S_{44}^*(s) e^{-[Q^*(\tau;b)-Q^*(s;b)]} \sinh[R^*(\tau;b) - R^*(s;b)] ds \\ &\quad - \int_0^\tau S_{34}^*(s) e^{-[Q^*(\tau;b)-Q^*(s;b)]} \cosh[R^*(\tau;b) - R^*(s;b)] ds, \\ S_{24}^*(\tau) &= \int_0^\tau S_{34}^*(s) e^{-[Q^*(\tau;b)-Q^*(s;b)]} \sinh[R^*(\tau;b) - R^*(s;b)] ds \\ &\quad - \int_0^\tau S_{44}^*(s) e^{-[Q^*(\tau;b)-Q^*(s;b)]} \cosh[R^*(\tau;b) - R^*(s;b)] ds. \end{aligned} \right.$$

For small τ we may provide approximations, using the following set of relations:

$$\left\{ \begin{aligned} (i) \quad &Q^*(\tau;c) - Q^*(s;c) \approx -H_{11}^*(\tau;c)(\tau - s), \quad c = a, b, \\ (ii) \quad &\sinh[R^*(\tau;c) - R^*(s;c)] \approx R^*(\tau;c) - R^*(s;c), \quad c = a, b, \\ (iii) \quad &\cosh[R^*(\tau;c) - R^*(s;c)] \approx 1 + \frac{1}{2}[R^*(\tau;c) - R^*(s;c)]^2, \quad c = a, b, \\ (iv) \quad &R^*(\tau;c) - R^*(s;c) \approx -H_{12}^*(\tau;c)(\tau - s), \quad c = a, b. \end{aligned} \right.$$

Applying (i) – (iv), we may obtain an approximation, for instance, to S_{13}^* , as follows:

$$\begin{aligned} S_{13}^*(\tau) &\approx - \int_0^\tau S_{34}^*(s) e^{H_{11}^*(\tau;a)(\tau-s)} H_{12}^*(\tau;a)(\tau - s) ds \\ &\quad - \int_0^\tau S_{33}^*(s) e^{H_{11}^*(\tau;a)(\tau-s)} \left\{ 1 + \frac{1}{2}[H_{12}^*(\tau;a)(\tau - s)]^2 \right\} ds. \end{aligned}$$

5. Summary and Conclusions

For a number of stochastic demand models, when production and inventory costs are quadratic, we have provided *via* the LQG approach, closed-form and approximate solutions for optimal production-inventory policies in a dynamic monopoly model. We found the following delimitations of our work: more work has to be done for multidimensional models when the demand vector consists of dependent components. The assumption about admissible controls

to be Lipschitz continuous in the initial state rules out “bang-bang” controls. No lower limit for the inventory level and the production rate are assumed. Finally, A , B , C , P , Q and M are all time-invariant matrices.

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