PROBABILISTIC GREEKS

Andrés D. Fundia*
Tecnológico de Monterrey, Campus Ciudad de México
Francisco Venegas Martínez
Tecnológico de Monterrey, Campus Ciudad de México

(Received 23 June 2004, accepted 27 August 2004)

Abstract

In the Black-Scholes-Merton option pricing formulas the coefficients that multiply the main variables (the price of the underlying and the strike price) are equal to some "Greeks" (partial derivatives of the price with respect to the main variables). In this paper we prove that this property is not only true for a log-normal distribution, but it is also satisfied by any distribution that comply with some natural conditions and by some exotic options. These identities are derived from a new integral representation of the Greeks. This representation allows to derive Greeks in an easy and systematic way simplifying the long computation of partial derivatives traditionally involved in obtaining them. When computing Greeks, these results can be applied to simplify the derivation of closed form expressions, to speed up numerical methods, and to obtain better accuracy.

Resumen

En la fórmula de valuación de opciones Black-Scholes-Merton, los coeficientes que multiplican a las principales variables (el precio del subyacente y el precio de ejercicio) son iguales a algunas "Griegas" (derivadas parciales del precio con respecto a las principales variables). En este trabajo probamos que esta propiedad no es sólo verdadera para una distribución log-normal, sino también se satisface para cualquier distribución que cumpla con algunas condiciones naturales y para algunas opciones exóticas. Estas identidades son derivadas a partir de una nueva representación integral de las Griegas. Esta representación permite determinar las Griegas en una forma sencilla y sistemática simplificando las complejidades computacionales y matemáticas tradicionalmente relacionadas en estos calculos. Cuando calculamos las griegas, estos resultados pueden ser aplicados para simplificar la derivación de expresiones cerradas, acelerar los métodos numéricos y obtener mejores ajustes.

JEL classification: C13, G12
Keywords: Estimation, Asset Pricing

* Departamento de Contabilidad y Finanzas, Tecnológico de Monterrey, Campus Ciudad de México. Calle del Puente 222, Oficinas 3, Segundo piso, Col. Ejidos de Huipulco, Del. Tlalpan, 14380, México, D. F., Teléfono: +52(55) 5483 2240. E-mail: andres.fundia@itesm.mx

1. Introduction

We will use the following standard notation: $S(t)$ or S_t denote the price of a financial asset at time $t \in [0, T]$, q is the continuously compounded rate of dividends that a holder of this asset receives, $C(t)$ denotes the time t value of a call option whose payoff function at expiration T is $(S(T)-K)^+ = \max\{S(T)-K, 0\}$, where K is the exercise price. Similarly, $P(t)$ denotes the time t value of a put option whose payoff function at expiration T is $(K-S(T))^+ = \max\{K-S(T), 0\}$. More generally, $V(t)$ denotes the time t value of a closed portfolio (i.e., such that no value is added or taken from it). Also, r is the continuously compounded risk free rate of interest.

The theory of option pricing (see Karatzas, I. and S. Shreve (1998) or Lamberton, D. and B. Lapeyere (1997)) establishes that in an arbitrage free market, under some standard assumptions:

$$V(t) = \mathrm{E}_t[V(T)e^{-r(T-t)}] \tag{1}$$

for a given probability distribution and family of sigma algebras $F_t, 0 \leq t \leq T$. Where, E_t and Pr_t denote the expectation and probabilities taken with this distribution, and conditioning on F_t. We will use the usual name Risk Neutral Valuation for prices based on formula (1).

Many closed form formulas for European options can be presented as:

$$C(t) = c_S S(t) + c_K K, \tag{2}$$

$$P(t) = p_K K + p_S S(t). \tag{3}$$

We will refer to c_S, c_K, p_S and p_K as the coefficients of the corresponding formula. This is a slight abuse of notation since they themselves depend on $S(t)$ and K. One single coefficient is undefined, but the pair c_S, c_K is defined as two values that substituted in equation (2) give the value of the call (similarly for puts).

Let $A = \{w : S(T, w) \geq K\}$, let A^c be its complement, and let 1_A and 1_{A^c} be their characteristic functions. Then, these coefficients admit the following representation:

$$c_S = \mathrm{E}_t \left[\frac{S(T)e^{-r(T-t)}}{S(t)} 1_A \right], \tag{4}$$

$$c_K = -\mathrm{E}_t \left[e^{-r(T-t)} 1_A \right], \tag{5}$$

$$p_S = -\mathrm{E}_t \left[\frac{S(T)e^{-r(T-t)}}{S(0)} 1_A \right], \tag{6}$$

$$p_K = \mathrm{E}_t \left[e^{-r(T-t)} 1_A \right]. \tag{7}$$

If interest rates are assumed to be deterministic (a frequently used approximation for derivatives on stock and foreign exchange markets), formulas (5) and (7) can be expressed as $-e^{-r(T-t)}\mathrm{Pr}_t[A]$ and $e^{-r(T-t)}\mathrm{Pr}_t[A^c]$. We keep the more general expressions because they are also valid when interest rates are stochastic.

These identities hold under any model where derivatives can be valued using the Risk Neutral Universe (i.e., where equation (1) is satisfied). Many variations of them are well known, some applications and generalizations can be found in Crouhy, M. *et al.* (2000), Geman, H. *et al.* (1995) and Gerber, H. and E. Shiu (1996). Since they are crucial for the results presented here, a proof is provided in appendix A.

Consider, now, the Black-Scholes-Merton model (see Black, F. and M. Scholes (1973) and Merton, R. (1973)). We will refer to it as B-S-M. Finding the expectations (4), (5), (6) and (7) under this model the B-S-M equations are obtained:

$$C(t) = S(t)e^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{8}$$

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S(t)e^{-q(T-t)}N(-d_1), \tag{9}$$

$$d_1 = \frac{\ln(\frac{S(t)}{K}) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}},$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}.$$

Where $N(.)$ is the cumulative distribution function of the standard normal distribution.

Consider now $\Delta_{call}(t) = \partial C(t)/\partial S(t)$. In the B-S-M model, after a laborious differentiation of formula (8) it turns out that $\Delta_{call} = e^{-q(T-t)}N(d_1)$, *i.e.* Δ_{call} equals the coefficient of the price. We prove here that the identity between Δ_{call} and the coefficient c_S (or the corresponding expectation) holds under any distribution that satisfies a very natural condition. Similar results are shown for puts, some exotic options, and other Greeks.

The key property is that, while the random variable $\frac{S(T)e^{-r(T-t)}}{S(t)}$ depends on σ and T, it is independent from the values of $S(t)$, K and r. So, the conditions required are:

$$\frac{S(T)e^{-r(T-t)}}{S_t} \text{ is independent of } S_t, \tag{10}$$

$$\frac{S(T)e^{-r(T-t)}}{S_t} \text{ is independent of } K, \tag{11}$$

$$\frac{S(T)e^{-r(T-t)}}{S_t} \text{ is independent of } r. \tag{12}$$

Condition (10) only says that we are interested in models of $S(T)$ that consider proportional growth rather than absolute growth, a necessary condition to model financial assets. Condition (11) says that the asset growth is independent of the strike price of an option written on it. This is satisfied in standard options where the strike price is fixed. Condition (12) says that, once discounted, the asset price is independent of the interest rates. This is true in the Risk Neutral Universe (see Hull, J. (2000)).

These conditions are applied independently of each other to obtain equalities between Greeks and expectations. The fact that Greeks equal the corresponding coefficients has often been cited as a rule of thumb to remember the Greeks. This rule gets fully explained here. Its explanation is provided by lemma 1, below, and the fact that the first summand in the left hand side of the lemma vanishes for most financial derivatives.

The equality between coefficients and Greeks follows from a new integral representation of derivatives prices (proposition 1, below). Even though it is simple to obtain, it is very powerful. It allows to derive Greeks in an easy and systematic way simplifying the long computation of partial derivatives traditionally involved in obtaining them.

In section 2, the main results are stated. Section 3 provides some technical lemmas needed to obtain these results. In section 4, the results are proven.

2. Greeks as Expectations

In this section we prove the identities between Δ_{call}, κ_{call}, and ρ_{call} (derivatives of the call price with respect to S_t, K, and r) and the corresponding expectations. We need only to assume conditions (10), (11) and (12) respectively. We also state similar identities for Δ_{put}, κ_{put}, and ρ_{put}. To prove them the same procedure, with minor modifications, can be followed. Alternatively, the results for puts can be deduced from the results for calls using the put-call parity.

Theorem 1

- If condition (10) is satisfied, then

$$\Delta_{call} = \mathrm{E}_t[\frac{S_T\ e^{-r(T-t)}}{S_t} 1_A],$$
$$\Delta_{put} = -\mathrm{E}_t[\frac{S_T\ e^{-r(T-t)}}{S_t} 1_{A^c}].$$

- If condition (11) is satisfied, then

$$\kappa_{call} = -\mathrm{E}_t[e^{-r(T-t)} 1_A],$$
$$\kappa_{put} = \mathrm{E}_t[e^{-r(T-t)} 1_{A^c}].$$

- If condition (12) is satisfied, then

$$\rho_{call} = (T-t)K\,\mathrm{E}_t[e^{-r(T-t)} 1_A],$$
$$\rho_{put} = -(T-t)K\,\mathrm{E}_t[e^{-r(T-t)} 1_{A^c}].$$

Using the representation of coefficients as expectations (equations (4), (5), (6) and (7)), this theorem can also be expressed in terms of the coefficients. Obtaining, thus, Corollary 1

- If conditions (10) and (11) are satisfied, then (2) is satisfied with $c_S = \Delta_{call}$ and $c_K = \kappa_{call}$ and (3) is satisfied with $p_S = \Delta_{put}$ and $p_K = \kappa_{put}$.
- If conditions (10) and (12) are satisfied, then (2) is satisfied with $c_S = \Delta_{call}$ and $c_K = -\rho_{call} K^{-1}(T-t)^{-1}$ (3) is satisfied with $p_S = \Delta_{put}$ and $p_K = -\rho_{put} K^{-1}(T-t)^{-1}$.

There is a simple and general probabilistic expression for $\Gamma_{call} = \partial^2 C(t)/\partial S(t)^2$ and for $\Gamma_{put} = \partial^2 P(t)/\partial S(t)^2$:

Theorem 2

- If condition (10) is satisfied, then

$$\Gamma_{call} = \frac{K e^{-r(T-t)}}{S_t} f_{S_T}(K),$$
$$\Gamma_{put} = \frac{K e^{-r(T-t)}}{S_t} f_{S_T}(K).$$

Where, as usual, $f_X(x)$ denotes the density function of the random variable X applied to x. Note that under the B-S-M model, using the corresponding log-normal distribution for f_{S_T}, and after some algebraic manipulations, the usual expression for Γ is obtained (see Wilmott, P (2000)). These results will be derived from the following representation of the price of a derivative. Its proof is provided in appendix B.

Proposition 1

Let

$$y_0 = \frac{K e^{-r(T-t)}}{S_t} \quad \text{and} \quad Y = \frac{S(T) e^{-r(T-t)}}{S_t},$$

then

$$C(t) = \int_{y_0}^{\infty} \left(S_t y - K e^{-r(T-t)}\right) \mathrm{d}F_Y(y), \tag{13}$$

$$P(t) = \int_{-\infty}^{y_0} \left(S_t y - K e^{-r(T-t)}\right) \mathrm{d}F_Y(y). \tag{14}$$

Where $F_X(x)$ denotes the cumulative distribution function of the random variable X, applied to x, and conditioning on F_t (*i.e.* assuming that S_t is known).

3. Calculus Lemmas

To compute Δ_{call}, we need to differentiate equation (13). Permuting the derivative and the integral is not possible because y_0 depends on S_0, K, and r. In this section we develop results that take care of this obstacle. For this, we will use the following Lemma (see Haaser, N. *et al.* (1964)).

Lemma 1

Let $f(x,y)$ be a continuous function on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$ such that $\frac{\partial f(x,y)}{\partial y}$ exists and is continuous. Let a and b be two differentiable functions on

$[\beta_1, \beta_2]$ such that for every $y \in [\beta_1, \beta_2]$, $a(y) \in [\alpha_1, \alpha_2]$ and $b(y) \in [\alpha_1, \alpha_2]$. Then,

$$\frac{\partial \left(\int_{a(y)}^{b(y)} f(x,y) \right)}{\partial y} \mathrm{d}x = \int_{a(y)}^{b(y)} \frac{\partial \left(f(x,y) \right)}{\partial y} \mathrm{d}x + f(b(y), y) b'(y) - f(a(y), y) a'(y). \tag{15}$$

In the applications, after a change of variables, $b(y)$ will be the strike price of a financial derivative and f the payoff of a derivative. Also, only one of the integration limits will depend on y. In this case the second and third term of the right hand side of equation (15) vanish. For ease of reference we state this simplified result.

Lemma 2

Assume that a, b and f satisfy the conditions of Lemma 1. If a is constant and $f(b(y), y) = 0$, then

$$\frac{\partial \int_{a}^{b(y)} f(x,y)}{\partial y} \mathrm{d}x = \int_{a}^{b(y)} \frac{\partial f(x,y)}{\partial y} \mathrm{d}x. \tag{16}$$

If b is constant and $f(a(y), y) = 0$, then

$$\frac{\partial \int_{a(y)}^{b} f(x,y)}{\partial y} \mathrm{d}x = \int_{a(y)}^{b} \frac{\partial f(x,y)}{\partial y} \mathrm{d}x. \tag{17}$$

4. Proofs of the Theorems

Proof of Theorem 1. We proceed now to compute Δ_{call} differentiating the integral of (13), applying lemma 2, and using condition (10):

$$\begin{aligned} \Delta_{call} &= \frac{\partial C(t)}{\partial S_t} \\ &= \frac{\partial}{\partial S_t} \left(\int_{y_0}^{\infty} (S_t y - K e^{-r(T-t)}) \mathrm{d}F_Y(y) \right) \\ &= \int_{y_0}^{\infty} \frac{\partial \left(S_t y - K e^{-r(T-t)} \right)}{\partial S_t} \mathrm{d}F_Y(y) \\ &= \int_{y_0}^{\infty} y \mathrm{d}F_Y(y) \\ &= \int_{K}^{\infty} \left(\frac{S_T e^{-r(T-t)}}{S_t} \right) \mathrm{d}F_{S_T}(S_T) \\ &= \mathrm{E}_t \left(\frac{S_T e^{-r(T-t)}}{S_t} 1_A \right). \end{aligned} \tag{18}$$

Similarly, we can obtain κ_{call} assuming (11).

$$
\begin{aligned}
\kappa_{call} &= \frac{\partial C(t)}{\partial K} \\
&= \frac{\partial}{\partial K}\left(\int_{y_0}^{\infty}(S_t y - K e^{-r(T-t)})\mathrm{d}F_Y(y)\right) \\
&= \int_{y_0}^{\infty}\frac{\partial(S_t y - K e^{-r(T-t)})}{\partial K}\mathrm{d}F_Y(y) \\
&= \int_{y_0}^{\infty} -e^{-r(T-t)}\mathrm{d}F_Y(y) \\
&= \int_{K}^{\infty} -e^{-r(T-t)}\mathrm{d}F_{S_T}(S_T) \\
&= -\mathrm{E}_t(e^{-r(T-t)}1_A).
\end{aligned}
$$

Assuming (12), we can also obtain ρ_{call}.

$$
\begin{aligned}
\rho_{call} &= \frac{\partial C(t)}{\partial r} \\
&= \frac{\partial}{\partial r}\left(\int_{y_0}^{\infty}(S_t y - K e^{-r(T-t)}\right)\mathrm{d}F_Y(y) \\
&= \int_{y_0}^{\infty}\frac{\partial(S_t y - K e^{-r(T-t)})}{\partial r}\mathrm{d}F_Y(y) \\
&= \int_{y_0}^{\infty} -(T-t)K e^{-r(T-t)}\mathrm{d}F_Y(y) \\
&= -(T-t)K\,\mathrm{E}_t(e^{-r(T-t)}1_A).
\end{aligned}
$$

Proof of Theorem 2. During the proof of theorem 1, in equation (18), we obtained the following equality $\Delta_{call} = \int_{y_0}^{\infty} y\mathrm{d}F_Y(y)$. So [1]

$$
\begin{aligned}
\Gamma_{call} &= \frac{\partial \Delta_{call}(t)}{\partial S_t} \\
&= \frac{\partial\left(\int_{y_0}^{\infty} y\mathrm{d}F_Y(y)\right)}{\partial S_t} \\
&= \frac{\partial\left(\int_{y_0}^{\infty} y f_Y(y)\mathrm{d}y\right)}{\partial S_t}.
\end{aligned}
$$

Since we are assuming condition (10), Y is independent of S_t. The integral of the last expression only depends on S_t through y_0, the limit of integration. So, in this case, we can just use the Fundamental Theorem of Calculus to obtain:

[1] In the proof, we are assuming that S_T follows a continuous distribution. The proof can also be done for a discrete distribution and is almost identical.

$$\begin{aligned}\Gamma_{call} &= y_0 f_Y(y_0)\frac{\mathrm{d}y_0}{\mathrm{d}S_t}\\ &= \frac{K\,e^{-r(T-t)}}{S_t} f_{S_T}(K).\end{aligned}$$

The last equality follows from the definition of y_0 and from the formula of change of variables for density functions.

5. Concluding Remarks

In this paper, we have proved that the partial derivatives of the option premium with respect to the price of the underlying and the strike price, main variables of Black-Scholes-Merton option pricing formulas, do not need the assumption of log-normal distribution to compute them, but they are also satisfied by any distribution that comply with some natural conditions. Then, we suggest a new way to derive Greeks from a new integral representation. This proposal follows two theorems, proven in the section 4, that allow us to reduce the computational time to obtain them. Finally, we show that our results fullfill the features of the original Greeks.

Appendices

A. Coefficients as Expectations

Proof of equalities (4), (5), (6), and (7). To apply (1), we need a closed portfolios, so if $S(t)$ is the price of an asset that pays dividends, let $\widehat{S}(t) = S(t)e^{qt}$ be the portfolio resulting from reinvesting all dividends in the asset. Defining also $\widehat{K} = K\,e^{qt}$, the payoff function of the call is $(S(T) - K)^+ = e^{-qT}(\widehat{S}(T) - \widehat{K})^+$. So the call on S can be treated as a call on $\widehat{S}$, with strike price $\widehat{K}$, multiplied by a constant. This is the usual method to treat dividends and all the extensions that follow. (See Hull, J. (2000) and Merton, R. (1973)). Then

$$\begin{aligned}C(t) &= \mathrm{E}_t(e^{-qT}(\widehat{S}(T) - \widehat{K})^+ e^{-r(T-t)})\\ &= \mathrm{E}_t((S(T) - K)^+ e^{-r(T-t)})\\ &= \mathrm{E}_t((S(T) - K)e^{-r(T-t)}1_A)\\ &= S(t)\mathrm{E}_t\left(\frac{S(T)e^{-r(T-t)}}{S(t)}1_A\right) - K\,\mathrm{E}_t(e^{-r(T-t)}1_A).\end{aligned}$$

The results follow from the last expression.

B. The Integral Representation

Proof of proposition (1). Starting from equation (1), we have

$$
\begin{aligned}
C(t) &= \mathrm{E}((S(T) - K)^{+} e^{-r(T-t)}) \\
&= \int_{0}^{\infty} (S_T - K)^{+} e^{-r(T-t)} \mathrm{d}F_{S_T}(S_T) \\
&= \int_{K}^{\infty} (S_T - K) e^{-r(T-t)} \mathrm{d}F_{S_T}(S_T) \\
&= \int_{K}^{\infty} \left(S_t \left(\frac{S_T e^{-r(T-t)}}{S_t} \right) - K e^{-r(T-t)} \right) \mathrm{d}F_{S_T}(S_T) \\
&= \int_{y_0}^{\infty} (S_t y - K e^{-r(T-t)}) \mathrm{d}F_Y(y).
\end{aligned}
$$

The last equality follows from a simple change of variables. The proof for the put is similar.

References

Black, F. and M. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81, pp. 637-654.

Broadie, M. and P. Glasserman (1996). Estimating Security Price Derivatives Using Simulation. *Management Science*, 42, pp. 269-285.

Fu, M. and J. Hu (1995). Sensitivity Analysis for Monte Carlo Simulation of Option Prices. *Probability in the Engineering and Informational Sciences*, 9, pp. 417-446.

Crouhy, M., D. Galai, and R. Mark (2000). Risk Management. McGraw-Hill.

Feller, W. (1973). Introduction to Probability Theory. Wiley.

Geman, H., N. El Karoui, and J. Rochet (1995). Changes of Numeraire, Changes of Probability Measure and Option Pricing. *Journal of Applied Probability*, 32.

Gerber, H. and E. Shiu (1996). Actuarial bridges to dynamic hedging and option pricing. *Insurance Mathematics and Economics*, 18, pp. 183-218.

Hull, J.(2000). Options Futures and other Derivatives. Prentice Hall, 5th edition.

Glasserman, P. (1991). Gradient Estimation Via Perturbation Analysis. Kluwer.

Haaser, N., J. LaSalle, and J. Sullivan (1964). Intermediate Analysis. Blaisdell.

Jarrow, R. and S. Turnbull (1996). Derivative Securities. South-Western.

Karatzas, I. and S. Shreve (1998). Methods of Mathematical Finance. Springer.

Lamberton, D. and B. Lapeyere (1997). Introduction to Stochastic Calculus Applied to Finance. Chapman & Hall.

Merton, R. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, 4, pp. 141-183.

Pellser, A. and T. Vorst (1994). The Binomial Model and the Greeks. *The Journal of Derivatives*, 1, pp. 45-49.

Wilmott, P. (2000). Quantitative Finance. John Wiley & sons.